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B. HOOGENBOOM

SPHERICAL FUNCTIONS AND DIFFERENTIAL OPERATORS ON COMPLEX GRASSMANN MANIFOLDS

kruislaan 413 1098 SJ amsterdam

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Spherical functions and differential operators on complex Grassmann manifolds

by

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ABSTRACT

Proofs are given of two theorems of Berezin and Karpelevič, which as far as we know never have been proved correctly. By using eigenfunctions of the Laplace-Beltrami operator it is shown that the spherical functions on a complex Grassmann manifold are given by a determinant of certain hypergeometric functions. By application of this result, it is proved that a certain system of operators, for which explicit expressions are given, generates the algebra of radial parts of invariant differential operators.

KEY WORDS & PHRASES: *Complex Grassmann manifold, spherical function, radial part of an invariant differential operator, hypergeometric function, Jacobi function*

0. INTRODUCTION AND MOTIVATION

In [1] BEREZIN and KARPELEVIČ gave an explicit expression for the zonal spherical functions on a complex Grassmann manifold. Unfortunately, no proof was given there.

In [9] TAKAHASI stated the same result, but he also gave a proof. This proof, however, was not correct. It relies upon another result of BEREZIN and KARPELEVIČ (also in [1], unproved), namely that the algebra $\delta(\mathbb{D}_0(G))$ of radial parts of invariant differential operators is generated by a system of operators Δ_i $(i = 1,\ldots,n)$, for which they could give explicit expressions. This being proved, it is sufficient to find the eigenfunctions of all Δ_i.

Takahasi's error was in the proof that $\delta(\mathbb{D}_0(G))$ is generated by the Δ_i. I'll try to indicate where he went wrong. He proceeded as follows.

Let $G := SU(n,n+k;\mathbb{C})$, and $\mathfrak{g} = \mathfrak{su}(n,n+k)$ its Lie algebra. Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $S(\mathfrak{p})$ be the symmetric algebra over $\mathfrak{p}$, and let $I(\mathfrak{p})$ be the subalgebra consisting of K-invariants. Let λ denote the canonical linear one-to-one mapping of $S(\mathfrak{g})$ onto $\mathbb{D}(G)$. Take $p \in I(\mathfrak{p})$. Then there exists a polynomial q such that $\delta(\lambda(p)) = q(s_1,\ldots,s_n)$ + terms of lower order. Define $p' := \delta(\lambda(p))-q(\Delta_1,\ldots,\Delta_n)$. Then we have degree p' < degree p. Now, according to Takahasi, the result follows by induction to the degree of p. But nothing guarantees us that p' has a highest order term with constant coefficients, so the induction step is not justified.

In this paper another proof of these two theorems is given, namely by using eigenfunctions of all $\delta(D)$ $(D \in \mathbb{D}_0(G))$ - say Φ - which have a certain convergent series expansion at ∞ in a positive Weyl chamber, instead of spherical functions - say ϕ - which are eigenfunctions of all $\delta(D)$ being regular in 0. To obtain these Φ, we only need to find the eigenfunctions of $\delta(\Omega)$ (radial part of the Laplace-Beltrami operator) which have the desired series expansion. That such a function is an eigenfunction of all $\delta(D)$ $(D \in \mathbb{D}_0(G))$ is a result of HARISH-CHANDRA [3]. A simpler proof is given by HELGASON [4]. Then we use that a spherical function ϕ can be written as a combination of Φ's. This gives us the first theorem of Berezin and Karpelevič. Finally, in the last chapter the second theorem of Berezin and Karpelevič, which states that the algebra $\delta(\mathbb{D}_0(G))$ is generated by the

Δ_i $(i = 1,\dots,n)$, is proved.

1. THE GROUP $G = SU(n,n+k;\mathbb{C})$

Let $G = SU(n,n+k;\mathbb{C})$ be the group of all complex $(n+m)\times(n+m)$ matrices with determinant 1 $(m = n+k,\ k \geq 0)$, which leave invariant the hermitian form:

$$x_1\bar{x}_1+x_2\bar{x}_2+\dots+x_n\bar{x}_n-x_{n+1}\bar{x}_{n+1}-\dots-x_{n+m}\bar{x}_{n+m}.$$

Then G is a connected, semisimple Lie group with finite center (see TAKAHASHI [9]).

Let $\mathfrak{g} = \mathrm{lie}(G)$ be the Lie algebra of G. Then $\mathfrak{g} = \mathfrak{su}(n,n+k;C)$ and $\mathfrak{g}$ is a real, semisimple Lie algebra.

Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, with

$$\mathfrak{k} = \{\begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix} : u^* = -u, v^* = -v,\ u \in M_n(\mathbb{C}),\ v \in M_m(\mathbb{C})\}$$

$$\mathfrak{p} = \{\begin{pmatrix} 0 & x \\ x^* & 0\end{pmatrix} : x \in M_{n,m}(\mathbb{C})\}.$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra. We may choose for $\mathfrak{a}$ the set of all matrices of the form

$$H_T = \begin{pmatrix} O_{n\times n} & T & O_{n\times k} \\ T & & \\ O_{k\times n} & & O_{m\times m} \end{pmatrix}$$

where $O_{p\times q}$ denotes the $(p\times q)$-matrix with only zeros as entries, and $T = \mathrm{diag}(t_1,\dots,t_n)$ $(t_i \in \mathbb{R}$ for all i). Let $\alpha_i \in \mathfrak{a}^*$ $(i = 1,\dots,n)$ be defined by $\alpha_i(H_T) = t_i$. Then the roots of $(\mathfrak{g},\mathfrak{a})$ are given by $\pm\alpha_i$, $\pm 2\alpha_i$ $(1 \leq i \leq n)$ and $\pm(\alpha_i\pm\alpha_j)$ $(1 \leq i < j \leq n)$, with multiplicities $m_{\alpha_i} = 2k$, $m_{2\alpha_i} = 1$ and $m_{\alpha_i\pm\alpha_j} = 2$.

Let $a_T := \exp H_T$, and $A := \{a_T = \exp H_T : H_T \in \mathfrak{a}\}$.

On the root system we choose an ordering such that the positive Weyl chamber C^+ is given by the T with $t_1 > t_2 > \ldots > t_n > 0$. Then the positive roots are α_i, $2\alpha_i$ $(1 \le i \le n)$ and $\alpha_i \pm \alpha_j$ $(1 \le i < j \le n)$. The simple roots are $\alpha_1-\alpha_2, \alpha_2-\alpha_3, \ldots, \alpha_{n-1}-\alpha_n, \alpha_n$.

Let Σ be the set of all roots, and Σ^+ the set of all positive roots.

From now on we identify T and H_T.

Let $\rho := \frac{1}{2} \Sigma_{\alpha\in\Sigma^+} m_\alpha \alpha$.

Then $\rho(T) = \Sigma_{i=1}^n \rho_i t_i$, with $\rho_i = k+1+2(n-i)$.

Let $\Delta(a_T) := \Pi_{\alpha\in\Sigma^+} (e^{\alpha(T)} - e^{-\alpha(T)})^{m_\alpha}$.

Then we have:

$$\Delta = \sigma\omega^2, \quad \text{with} \quad \sigma(a_T) = 2^{n(2k+1)} \Pi_{i=1}^n (\mathrm{sh}^{2k} t_i \,\mathrm{sh} 2t_i),$$

$$\text{and} \quad \omega(a_T) = 2^{\frac{1}{2}n(n-1)} \Pi_{i<j} (\mathrm{ch} 2t_i - \mathrm{ch} 2t_j).$$

Let $\mathbb{D}(G)$ be the algebra of left G-invariant differential operators on G, and let $\mathbb{D}_0(G)$ be the subalgebra of $\mathbb{D}(G)$ of right K-invariant operators. If $D \in \mathbb{D}_0(G)$, let $\delta(D)$ denote the radial part of D.

As usual let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ denote the sets of all complex numbers, real numbers, integers, positive (non zero) integers and negative (non zero) integers, respectively.

2. RADIAL PART OF THE LAPLACE-BELTRAMI OPERATOR

Let $\delta(\Omega)$ denote the radial part of the Laplace-Beltrami operator. In [3] HARISH-CHANDRA proved the following lemma:

<u>LEMMA 2.1</u>. *Let $H_1, \ldots, H_\ell$ be a basis of $\mathfrak{a}$, and let $(g^{ij})_{1\le i,j\le \ell}$ denote the inverse of the matrix with elements $B(H_i,H_j)$ ($B(.,.)$ Killing form). Then*

$$\delta(\Omega) = \sum_{1\le i,j\le\ell} \Delta^{-1} g^{ij} H_i \circ \Delta H_j. \tag{2.1}$$

Take for H_i the matrix H_{T_i}, with $T_i = \mathrm{diag}(0,\ldots,0,1,0,\ldots,0)$ (with 1 on the i-th place). Then

$$B(H_i,H_j) = \mathrm{tr}(\mathrm{ad}H_i \; \mathrm{ad}H_j)$$

$$= \sum_{\beta\in\Sigma} m_\beta\ \beta(H_i)\beta(H_j)$$

$$= 4(k+2n)\ \delta_{ij}.$$

So formula (2.1) gives:

$$\delta(\Omega) = (4(k+2n))^{-1}\sum_{i=1}^{n} \omega^{-2}\sigma^{-1} \frac{\partial}{\partial t_i} (\omega^2\sigma \frac{\partial}{\partial t_i}).$$

(As a differential operator H_i corresponds with $\partial/\partial t_i$). Hence:

$$4(k+2n)\delta(\Omega) = \sum_i\left(\frac{\partial^2}{\partial t_i^2} + (2\omega^{-1}\frac{\partial\omega}{\partial t_i} + \sigma^{-1}\frac{\partial\sigma}{\partial t_i})\frac{\partial}{\partial t_i}\right)$$

$$= \sum_i \omega^{-1}(\frac{\partial^2}{\partial t_i^2} + \sigma^{-1}\frac{\partial\sigma}{\partial t_i}\frac{\partial}{\partial t_i}) \circ \omega$$

$$- \sum_i \omega^{-1}(\frac{\partial^2}{\partial t_i^2} + \sigma^{-1}\frac{\partial\sigma}{\partial t_i}\frac{\partial}{\partial t_i})\omega$$

$$= \omega^{-1}S_1(L_1,\ldots,L_n) \circ \omega - \omega^{-1}S_1(L_1,\ldots,L_n)\omega,$$

where we have defined

$$L_i := \frac{\partial^2}{\partial t_i^2} + 2(k\coth t_i + \coth 2t_i)\frac{\partial}{\partial t_i}$$

and

$$S_j(L_1,\ldots,L_n) := \text{the } j\text{-th elementary symmetric polynomial in } L_1,\ldots,L_n$$

(see [9]).

Now define

$$\Delta_j := \omega^{-1}S_j(L_1,\ldots,L_n) \circ \omega,$$

then we have, because of the relation $S_j(L_1,\ldots,L_n)\ \omega = c_j\omega$ (c_j defined by $\Sigma_{j=0}^{n} c_j\xi^{n-j} = \Pi_{i=0}^{n-1} (\xi+4i(i+k+1))$, see [9]):

$$(2.2) \qquad 4((k+2n)\delta(\Omega) = \Delta_1 - \sum_{i=1}^{n-1} 4i(i+k+1),$$

3. EIGENFUNCTIONS OF $\delta(\Omega)$

In this chapter we make use of the following lemma (see [4], ch.II, prop. 1.10). Let Λ be the root lattice, that is $\Lambda = \{z_1\beta_1+\ldots+z_n\beta_n :$ $\beta_i \in \Sigma$, β_i is simple, $z_i \in \mathbb{Z}^+ \cup (0)\}$. Let γ denote the natural isomorphism of $\mathbb{D}(X)$ onto $I(A)$ ($X = G/K$, A Lie group corresponding to $\mathfrak{a}$, $I(A)$ set of W-invariant polynomials on A, see [4], ch.II, theorem 1.2).

LEMMA 3.1. *The equation*

$$\delta(\Omega)u = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)u$$

has a unique solution on C^+ *of the form*

$$u(H) = \Phi_\lambda(\exp H) = \sum_{\mu\in\Lambda} \Gamma_\mu \exp((\sqrt{-1}\lambda-\rho-\mu)H)$$

with $\Gamma_0 = 1$. $u = \Phi_\lambda \circ \exp$ *is also a solution of the system of differential equations*

$$(3.1) \qquad \delta(D)u = \gamma(D)(\sqrt{-1}\lambda)u, \quad D \in \mathbb{D}_0(G).$$

In our case, the function Φ_λ of the lemma takes the form

$$(3.2) \qquad \Phi_\lambda(a_T) = e^{(\sqrt{-1}\lambda-\rho)(T)} \sum_{\mu\in\Lambda} \Gamma_\mu(\lambda)e^{-\mu(T)},$$

where

$$T \in C^+$$
$$\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathfrak{a}_C^*$$
$$\Gamma_0 \equiv 1,$$

in order to be an eigenfunction of all $\delta(D)$, $D \in \mathbb{D}_0(G)$.

So we have to solve

$$(\omega^{-1}S_1(L_1,\dots,L_n)\circ\omega)u = \mu u,$$

i.e.

$$S_1(L_1,\dots,L_n)(\omega u) = \mu(\omega u).$$

Let us try a solution $u(T)$ of the form

$$\omega(T)u(T) = v_1(L_1)\cdot\ldots\cdot v_n(t_n),$$

where v_i is a solution of the equation

$$L_i v_i = -(\lambda_i^2+(k+1)^2)v_i, \quad t_i > 0, \tag{3.3}$$

such that v_i is of the form

$$v_i(t_i) = e^{(\sqrt{-1}\lambda_i-(k+1))t_i}\sum_{n=0}^{\infty}\Gamma_n e^{-nt_i}, \quad \Gamma_0 = 1. \tag{3.4}$$

DEFINITION 3.1. Let $v_i(t_i)$ be a solution of (3.3), which is of the form (3.4). Then we define

$$\Phi_\lambda(a_T) := \frac{v_1(t_1)\cdot\ldots\cdot v_n(t_n)}{\omega(a_T)}.$$

THEOREM 1.

a. $\Phi_\lambda(a_T)$ *satisfies* $\delta(\Omega)\Phi_\lambda(a_T) = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)\Phi_\lambda(a_T)$.

b. $\Phi_\lambda(a_T)$ *has a series expansion* (3.2).

PROOF.

a. According to (2.2) we have

$$4(k+2n)\delta(\Omega)\Phi_\lambda(a_T) = (\Delta_1 - \sum_{i=0}^{n-1} 4i(i+k+1))\Phi_\lambda(a_T). \tag{3.5}$$

Because of the relation $B(H_i,H_j) = 4(k+2n)\delta_{ij}$, the inner product $\langle .,. \rangle$ is given by $\langle\xi,\eta\rangle = (4(k+2n))^{-1}\Sigma_{i=1}^{n}\xi_i\eta_i$, if $\xi = (\xi_1,\dots,\xi_n)$, $\eta = (\eta_1,\dots,\eta_n)$. Hence

$$\Delta_1\Phi_\lambda(a_T) = \omega^{-1}S_1(L_1,\ldots,L_n)\circ\omega(\omega^{-1}\prod_{i=1}^n v_i(t_i))$$

$$= \omega^{-1}(-(4(k+2n)\langle\lambda,\lambda\rangle + n(k+1)^2))\prod_{i=1}^n v_i(t_i)$$

$$= -(4(k+2n)\langle\lambda,\lambda\rangle + n(k+1)^2)\Phi_\lambda(a_T), \tag{3.6}$$

because of the relation $L_i v_j(t_j) = -(\lambda_j^2+(k+1)^2)v_j(t_j)\delta_{ij}$. Since $\rho_i = k+1+2(n-i)$ we have $4(k+2n)\langle\rho,\rho\rangle = n(k+1)^2 + \Sigma_{j=0}^{n-1} 4j(k+1+j)$, and this together with (3.5) and (3.6) proves a.

b. To prove that $\Phi_\lambda(T)$ has a series of expansion (3.2) we use the fact that $v_i(t_i)$ is of the form (3.4). We have

$$\Phi_\lambda(a_T) = \frac{v_1(t_1)\cdot\ldots\cdot v_n(t_n)}{\omega(a_T)}.$$

According to (3.4) the numerator is of the form

$$e^{(\sqrt{-1}\lambda_1-(k+1))t_1+\ldots+(\sqrt{-1}\lambda_n-(k+1))t_n}\sum_{\ell_1=0}^\infty \Gamma_{\ell_1} e^{-\ell_1 t_1}\cdot\ldots\cdot\sum_{\ell_n=0}^\infty \Gamma_{\ell_n} e^{-\ell_n t_n}. \tag{3.7}$$

For the denominator we have

$$\omega(a_T) = 2^{\frac{1}{2}n(n-1)}\prod_{i<j}\tfrac{1}{2}(e^{2t_i}+e^{-2t_i}-e^{2t_j}-e^{-2t_j})$$

$$= 2^{2(n-1)t_1+2(n-2)t_2+\ldots+2t_{n-1}}\prod_{i<j}(1-e^{-2(t_i-t_j)})(1-e^{-2(t_i+t_j)}). \tag{3.8}$$

In C^+ we have $t_1 > t_2 > \ldots > t_n > 0$, so for all $T\in C^+$ the exponents in the denominator (i.e. $-2(t_i-t_j)$ and $-2(t_i+t_j)$ with $i<j$) are <0, so we have the power series expansions

$$\frac{1}{1-e^{-2(t_i-t_j)}} = \sum_{p=0}^\infty e^{-2p(t_i-t_j)},$$

$$\frac{1}{1-e^{-2(t_i+t_j)}} = \sum_{q=0}^\infty e^{-2q(t_i+t_j)}.$$

Using these power series expansion and formulas (3.7) and (3.8) we get for Φ_λ:

$$\Phi_\lambda(a_T) = e^{(\sqrt{-1}\lambda_1-(k+1)-2(n-1))t_1+\ldots+(\sqrt{-1}\lambda_{n-1}-(k+1)-2)t_{n-1}+(\sqrt{-1}\lambda_n-(k+1))t_n}.$$

$$\cdot \prod_{i=1}^n\left(\sum_{\ell_i=0}^\infty \Gamma_{\ell_i} e^{-\ell_i t_i}\right) \prod_{i<j}\left(\sum_{p=0}^\infty e^{-2p(t_i-t_j)} \sum_{q=0}^\infty e^{-2q(t_i+t_j)}\right),$$

i.e. $e^{(\sqrt{-1}\lambda-\rho)(T)}$ multiplied with a finite product of convergent series of the form $\sum_{\mu\in\Lambda} b_\mu(\lambda)e^{-\mu(T)}$.

Hence multiplication of the power series gives

$$\Phi_\lambda(a_T) = e^{(\sqrt{-1}\lambda-\rho)(T)} \sum_{\mu\in\Lambda} \Gamma_\mu(\lambda)e^{-\mu(T)}.$$

Clearly we have $\Gamma_0 \equiv 1$ which proves b. □

Now we've come to the point where we have to find the function $v_i(t_i)$ which satisfies (3.3) and (3.4). The equation $L_i v_i = \mu_i v_i$ can be seen as a differential equation for Jacobi functions (see [8]). The general equation for Jacobi functions is:

(3.9) $$(\Delta_{\alpha,\beta}(t))^{-1} \frac{d}{dt}\{\Delta_{\alpha,\beta}(t) \frac{du(t)}{dt}\} = -(\lambda^2+(\alpha+\beta+1)^2)u(t),$$

where $\Delta_{\alpha,\beta}(t) = (e^t-e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}$.

The left-hand side of (3.9) in the case $\alpha = k$, $\beta = 0$, $t = t_i$ is easily seen to be equal to $L_i u$. So let us try to find a solution of

(3.10) $$(\Delta_{k,0}(t_i))^{-1} \frac{\partial}{\partial t_i}\{\Delta_{k,0}(t_i) \frac{\partial u}{\partial t_i}\} = -(\lambda_i^2+(k+1)^2)u,$$

which is of the form (3.4).

Substitute $t_i := -\mathrm{sh}^2 t_i$. Then equation (3.10) leads to a hypergeometrical differential equation. If we let $t_i \to \infty$, (3.4) gives the asymptotic behaviour:

(3.11) $$v_i(t_i) = e^{(\sqrt{-1}\lambda_i-(k+1))t_i}(1+o(1)).$$

According to [2, 2.9(9)] the Jacobi function of the second kind

$$\Phi_{\lambda_i}^{(k,0)}(t_i) = (e^{t_i}-e^{-t_i})^{\sqrt{-1}\lambda_i-(k+1)} {}_2F_1(\tfrac12(-k+1-\sqrt{-1}\lambda_i),\tfrac12(k+1-\sqrt{-1}\lambda_i);\; 1-\sqrt{-1}\lambda_i;-\mathrm{sh}^{-2}t_i)$$

is a solution of (3.10) for all λ_i with $\mathrm{Im}\,\lambda_i \notin \mathbb{Z}^-$, having the asymptotic behaviour (3.11).

<u>LEMMA 3.2</u>. *$\Phi_{\lambda_i}^{(k,0)}(t_i)$ has a convergent series expansion* (3.4) *for* $t_i > 0$.

<u>PROOF</u>.

$$\Phi_{\lambda_i}^{(k,0)}(t_i) = (e^{t_i}-e^{-t_i})^{\sqrt{-1}\lambda_i-(k+1)} {}_2F_1(\tfrac12(-k+1-\sqrt{-1}\lambda_i),\tfrac12(k+1-\sqrt{-1}\lambda_i);\; 1-\sqrt{-1}\lambda_i;-\mathrm{sh}^{-2}t_i)$$

$$= (e^{t_i}+e^{-t_i})^{\sqrt{-1}\lambda_i-(k+1)} {}_2F_1(\tfrac12(k+1-\sqrt{-1}\lambda_i);\tfrac12(k+1-\sqrt{-1}\lambda_i);\; 1-\sqrt{-1}\lambda_i;\mathrm{ch}^{-2}t_i) \qquad \text{(see [2, 2.10(6)])}$$

$$= e^{(\sqrt{-1}\lambda_i-(k+1))t_i}(1+e^{-2t_i})^{\sqrt{-1}\lambda_i-(k+1)} \sum_{n=0}^{\infty} \frac{((\tfrac12(k+1-\sqrt{-1}\lambda_i))_n)^2}{(1-\sqrt{-1}\lambda_i)_n n!}(\mathrm{ch}^{-2}t_i)^n,$$

absolutely convergent for $t > 0$ since $0 < \mathrm{ch}^{-2}t_i < 1$. Hence

$$\Phi_{\lambda_i}^{(k,0)}(t_i) = e^{(\sqrt{-1}\lambda_i-(k+1))t_i}\sum_{n=0}^{\infty} \gamma_n e^{-2nt_i}(1+e^{-2t_i})^{-2n+\sqrt{-1}\lambda_i-k-1}.$$

The lemma follows by expansion of $(1+e^{-2t_i})^{-2n+\sqrt{-1}\lambda_i-k-1}$ in powers of e^{-2t_i}. □

Combining theorem 1, lemma 3.1 and lemma 3.2 we get

<u>THEOREM 2</u>. *The function*

$$\Phi_\lambda(a_T) = \frac{\Phi_{\lambda_1}^{(k,0)}(t_1)\cdot\ldots\cdot\Phi_{\lambda_n}^{(k,0)}(t_n)}{\omega(a_T)}$$

satisfies

$$\delta(D)\Phi_\lambda(a_T) = \gamma(D)(\sqrt{-1}\lambda)\Phi_\lambda(a_T)$$

for all $D \in \mathbb{D}_0(G)$.

4. SPHERICAL FUNCTIONS ON $SU(n,n+k;\mathbb{C})$

Let ϕ_λ be a spherical function on G, that is an eigenfunction of all $D \in \mathbb{D}_0(G)$, having value 1 at e. Then we have (see [5]):

$$\phi_\lambda(a_T) = \sum_{s\in W} c(s\lambda)\Phi_{s\lambda}(a_T), \qquad T \in C^+, \tag{4.1}$$

where W is the Weyl group of G and $\Phi_\lambda(a_T)$ an eigenfunction of $\delta(\Omega)$ with a series expansion (3.2). Our main goal in this chapter is to find ϕ_λ, or to find the function c.

Let us first look at the rank 1 case (see [8]). As a solution of the hypergeometrical differential equation (3.10), which is regular for $t = 0$, we get:

$$\phi_{\lambda_i}^{(k,0)}(t_i) = {}_2F_1(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i);\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);k+1;-\mathrm{sh}^2 t_i).$$

Now, assume that $\lambda_i \notin \sqrt{-1}\mathbb{Z}$. Then we know from [2, 2.10(2)] that

$$\begin{aligned} {}_2F_1(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i),\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);k+1;\mathrm{sh}^2 t_i) = \\ \sum_{s\in\{1,-1\}} c(s\lambda_i)(e^{t_i}-e^{-t_i})^{\sqrt{-1}s\lambda_i-(k+1)} \quad . \\ \cdot\ {}_2F_1(\tfrac{1}{2}(-k+1-\sqrt{-1}s\lambda_i),\tfrac{1}{2}(k+1-\sqrt{-1}s\lambda_i);1-\sqrt{-1}s\lambda_i;-\mathrm{sh}^{-2}t_i) \end{aligned}$$

with

$$c(\lambda_i) = \frac{\Gamma(k+1)\Gamma(-\sqrt{-1}\lambda_i)2^{\sqrt{-1}\lambda_i+k+1}}{\Gamma(\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i))\Gamma(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i))}. \tag{4.2}$$

So we have

(4.3) $$\phi_{\lambda_i}(t_j) = c(\lambda_i)\Phi_{\lambda_i}(t_j) + c(-\lambda_i)\Phi_{-\lambda_i}(t_j)$$

(from now on we omit the indices $(k,0)$, that is we'll write ϕ_{λ_i} instead of $\phi_{\lambda_i}^{(k,0)}$ etc.) where c is defined as in (4.2). Because $(-\lambda_i)^2 = \lambda_i^2$ the following relation is also valid.

(4.4) $$L_i\phi_{\lambda_i}(t_j) = -(\lambda_i^2+(k+1)^2)\phi_{\lambda_i}(t_j).$$

<u>DEFINITION 4.1.</u>

$$\phi_\lambda(a_T) := \frac{A}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \cdot \frac{\det(\phi_{\lambda_i}(t_j))_{1\le i,j\le n}}{\omega(a_T)} .$$

(A is a normalization constant, independent of T and λ, which has yet to be determined).

We want to prove that $\phi_\lambda(a_T)$ is a spherical function on G. Therefore, we'd like to write ϕ_λ as a combination of Φ_λ's, in a way which is similar to (4.1). According to [9] we have $W = \{s:s(t_1,\dots,t_n) = (\varepsilon_1 t_{\sigma(\ )}, \dots,\varepsilon_n t_{\sigma(n)}),\ \varepsilon_i = \pm 1,\ \sigma \in S_n\}$. We'll denote such a $s \in W$ by $s = (\varepsilon,\sigma)$ with $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)$ and $\sigma \in S_n$. Thus

$$A^{-1}.\omega(a_T)\phi_\lambda(a_T) = \frac{\det(\phi_{\lambda_i}(t_j))}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)}$$

$$= \frac{\sum_{\sigma\in S_n}(-1)^{\operatorname{sgn}\sigma}\prod_{p=1}^n\phi_{\lambda_{\sigma(p)}}(t_p)}{(-1)^{\frac12 n(n-1)}\det((\lambda_i^2)^{j-1})}$$

$$= \frac{\sum_{\sigma\in S_n}(-1)^{\operatorname{sgn}\sigma}\sum_{\substack{\varepsilon_i=\pm1\\ i=1,\dots,n}} c(\varepsilon_1\lambda_{\sigma(1)})\Phi_{\varepsilon_1\lambda_{\sigma(1)}}(t_1)\cdot\ldots\cdot c(\varepsilon_n\lambda_{\sigma(n)})\Phi_{\varepsilon_n\lambda_{\sigma(n)}}(t_n)}{(-1)^{\frac12 n(n-1)}\det((\lambda_i^2)^{j-1})}$$

$$= \sum_{\substack{\sigma\in S_n\\ \varepsilon_i=\pm1}} \frac{c(\varepsilon_1\lambda_{\sigma(1)})\cdot\ldots\cdot c(\varepsilon_n\lambda_{\sigma(n)})}{(-1)^{\frac12 n(n-1)}\det((\varepsilon_1\lambda_{\sigma(i)})^{2(j-1)})}\prod_{p=1}^n\Phi_{\varepsilon_p\lambda_{\sigma(p)}}(t_p).$$

Hence

$$(4.5) \qquad \phi_\lambda(a_T) = \sum_{s\in W} C(s\lambda)\Phi_{s\lambda}(a_T),$$

where

$$(4.6) \qquad C(\lambda) = A.\frac{c(\lambda_1)\cdot\ldots\cdot c(\lambda_n)}{(-1)^{\frac{1}{2}n(n-1)}\det(\lambda_i^{2(j-1)})}$$

Since $\langle s\lambda,s\lambda\rangle = \langle\lambda,\lambda\rangle$ for all $s \in W$, it follows from (4.5) and theorem 1 a that

$$(4.7) \qquad \delta(\Omega)\phi_\lambda(a_T) = -(\langle\lambda,\lambda\rangle+\langle\rho,\rho\rangle)\phi_\lambda(a_T).$$

LEMMA 4.1. (HUA [6]) *Suppose* $f_1(x),\ldots,f_n(x)$ *are* C^∞*-functions on a real interval* I. *Let*

$$F(x_1,\ldots,x_n) := \frac{\det(f_i(x_j))}{\prod_{i<j}(x_i-x_j)}.$$

Then F *is* C^∞ *and symmetric on* I^n *and, for* $a \in I$,

$$F(a,\ldots,a) = \frac{(-1)^{\frac{1}{2}n(n-1)}}{1!2!\ldots(n-1)!}\det(f_i^{(j-1)}(a)).$$

Moreover, if all the f_i *are polynomials, then so is* F.

PROOF. (Sketch) Use complete induction with respect to n, by writing

$$\det(f_i(x_j)) = (x_2-x_1)\ldots(x_n-x_1).\det\begin{vmatrix} f_1(x_1) & \ldots & f_n(x_1) \\ \frac{f_1(x_2)-f_1(x_1)}{x_2-x_1} & \ldots & \frac{f_n(x_2)-f_n(x_1)}{x_2-x_1} \\ \vdots & & \\ \frac{f_1(x_n)-f_1(x_1)}{x_n-x_1} & \ldots & \frac{f_n(x_n)-f_n(x_1)}{x_n-x_1} \end{vmatrix}$$

and next expanding the determinant with respect to the first row. □

According to [2, 2.8(20)], we have

$$(4.8)\qquad \frac{d^\ell}{dz^\ell}\, {}_2F_1(a,b;c;z) = \frac{(a)_\ell (b)_\ell}{(c)_\ell}\, {}_2F_1(a+\ell,b+\ell;c+\ell;z).$$

Now

$$\lim_{T\to 0} \frac{\det(\phi_{\lambda_i})(t_j))}{\omega(T)} = \lim_{T\to 0} \frac{\det({}_2F_1(\tfrac12(k+1+\sqrt{-1}\lambda_i),\tfrac12(k+1-\sqrt{-1}\lambda_i);k+1;-\mathrm{sh}^2 t_j)}{2^{n(n-1)} \prod_{i<j} (\mathrm{sh}^2 t_i - \mathrm{sh}^2 t_j)}.$$

Using lemma 4.1 and (4.8) we see that this expression is equal to

$$\frac{2^{-n(n-1)}(-1)^{\frac12 n(n-1)}}{1!2!\ldots(n-1)!} \det \begin{vmatrix} 1 & \cdots & 1 \\ -\tfrac14(\frac{1}{k+1})(\lambda_1^2+(k+1)^2) & \cdots & -\tfrac14(\frac{1}{k+1})(\lambda_n^2+(k+1)^2) \\ \vdots & & \vdots \\ (-\tfrac14)^{n-1}(\frac{1}{(k+1)\ldots(k+(n-1))})(\lambda_1^2+(k+1)^2)\ldots(\lambda_1^2+(k+2n-1)^2) & \cdots & \end{vmatrix}$$

$$= \frac{1}{2^{2n(n-1)}\prod_{j=1}^{n-1}\{(k+j)^{n-j} j!\}} \det \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1^2+(k+1)^2 & \cdots & \lambda_n^2+(k+1)^2 \\ \vdots & & \vdots \\ (\lambda_1^2+(k+1)^2)^{n-1} & \cdots & (\lambda_n^2+(k+1)^2)^{n-1} \end{vmatrix}$$

$$= \frac{(-1)^{\frac12 n(n-1)}}{2^{2n(n-1)}\prod_{j=1}^{n-1}\{(k+j)^{n-j} j!\}} \prod_{i<j} (\lambda_i^2-\lambda_j^2).$$

Hence, if we take

$$(4.9)\qquad A = (-1)^{\frac12 n(n-1)}\, 2^{2n(n-1)} \prod_{j=1}^{n-1} \{(k+j)^{n-j} j!\}$$

in definition 3.1 we obtain

$$(4.10) \qquad \phi_\lambda(a_0) = 1.$$

Now, since it is obvious from the definition that ϕ_λ is W-invariant and C^∞ everywhere on A, it follows from theorem 2 and the relations (4.5) and (4.10) that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\lambda_p \notin \sqrt{-1}\mathbb{Z}$ for all $p, \phi_\lambda(a_T)$ is the restriction to A of a spherical function on G. Because the set $\{\lambda \in \mathbb{C}^n: \sqrt{-1}\lambda_p \notin \mathbb{Z}\ \forall p\}$ is an open, dense subset of $\mathbb{C}^n$, we can catch all λ by analytic continuation (if $\lambda_p = \lambda_q$ for some p,q, p≠q continuation according to lemma 4.1), so we have proved the first theorem of Berezin and Karpelevič.

<u>THEOREM 3</u>. (BEREZIN and KARPELEVIČ [1]). *The zonal spherical functions* ϕ_λ *on* $G = SU(n,n+k;\mathbb{C})$ *are given by*

$$\phi_\lambda(a_T) = \frac{A}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \cdot \frac{\det\left({}_2F_1(\tfrac{1}{2}(k+1+\sqrt{-1}\lambda_i),\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i);k+1;-sh^2 t_j)\right)}{2^{\frac{1}{2}n(n-1)} \prod_{i<j}(ch2t_i-ch2t_j)}$$

where A *is as in* (4.9).

5. THE ALGEBRA $\delta(\mathbb{D}_0(G))$

Now we come to the point where we can prove the second theorem of Berezin and Karpelevič. We proceed as follows. First, we show that the functions ϕ_λ satisfy $\Delta_j\phi_\lambda = a_j(\lambda)\phi_\lambda$ for all j, and next, by using a method of KOORNWINDER (see [7],§6), we show that every differential operator, which has all the ϕ_λ as eigenfunctions, is a polynomial in the Δ_j $(j = 1,\ldots,n)$, and this polynomial is uniquely determined. Thus, because of the fact that $\delta(D)\phi_\lambda = \gamma(D)(\sqrt{-1}\lambda)\phi_\lambda$ $(D \in \mathbb{D}_0(G))$ (this follows from theorem 2 and (4.5)) it follows that the algebra $\delta(\mathbb{D}_0(G))$ is generated by the Δ_j $(j = 1,\ldots,n)$.

For reasons of convenience we'll work with a slightly larger set than $\delta(\mathbb{D}_0(G))$.

<u>LEMMA 5.1</u>. $\Delta_j\phi_\lambda(a_T) = a_j(\lambda)\phi_\lambda(a_T)$ *for all* j.

<u>PROOF</u>. In 1 variable t we have

$$L_i\Phi_{\lambda_j}(t) = -(\lambda_j^2+(k+1)^2)\Phi_{\lambda_j}(t)\delta_{ij}.$$

Hence

$$\prod_{i=1}^{n}(\xi+L_i)\prod_{j=1}^{n}\Phi_{\lambda_j}(t_j) = \prod_{i=1}^{n}(\xi-(\lambda_i^2+(k+1)^2)\prod_{j=1}^{n}\Phi_{\lambda_j}(t_j).$$

Define on $\mathfrak{a}_{\mathbb{C}}^*$ the functions $a_j(\lambda)$ by

$$\prod_{i=1}^{n}(\xi-(\lambda_i^2+(k+1)^2)) = \sum_{j=0}^{n} a_j(\lambda)\xi^{n-j}.$$

Then

$$S_j(L_1,\dots,L_n)\prod_{i=1}^{n}\Phi_{\lambda_i}(t_i) = a_j(\lambda)\prod_{i=1}^{n}\Phi_{\lambda_i}(t_i) \quad \text{for all } j.$$

$$\Rightarrow (\omega^{-1}S_j(L_1,\dots,L_n)\circ\omega)\Phi_\lambda(a_T) = a_j(\lambda)\Phi_\lambda(a_T) \quad \text{for all } j.$$

$$\Rightarrow (\omega^{-1}S_j(L_1,\dots,L_n)\circ\omega)\phi_\lambda(a_T) = a_j(\lambda)\phi_\lambda(a_T) \quad \text{for all } j.$$

$$\Rightarrow \Delta_j\phi_\lambda(a_T) = a_j(\lambda)\phi_\lambda(a_T) \quad \text{for all } j. \quad \square$$

For the second part: remark first that every differential operator which is a polynomial in the Δ_j, has to have all ϕ_λ as eigenfunctions, because of lemma 5.1. So we have to prove that every D which has all ϕ_λ as eigenfunctions must be a polynomial in the Δ_j. We'll restrict ourself to those ϕ_λ which are polynomials, that is $\frac{1}{2}(k+1\pm\sqrt{-1}\lambda_i) \in \mathbb{Z}^-$. If we can prove that this, i.e. every D which has all polynomial ϕ_λ as eigenfunctions, is a polynomial in the Δ_j, we are done because of the remark above.

Let N be the ordered set of all n-tuples $\mu = (\mu_1,\dots,\mu_n)$ with $\mu_i \in \mathbb{Z}$ for all i, and $\mu_1 \geq \mu_2 \geq\dots\geq \mu_n \geq 0$, and let $<$ denote the lexicographical ordering on N.

Let $t = (t_1,\dots,t_n)$ with $t_i \in \mathbb{Z}$ for all i.

Now, let $\phi_{\lambda_i}(t)$ be a polynomial. Say

$$\tfrac{1}{2}(k+1-\sqrt{-1}\lambda_i) = -m_i-n+i \qquad \text{for } i = 1,\dots,n \quad \text{and } m \in N.$$

Then $\phi_{\lambda_i}(t)$ becomes

$$\phi_{\lambda_i}(t) = {}_2F_1(-(m_i+n-i),\ m_i+n-i+k+1;k+1;-\mathrm{sh}^2 t).$$

We'll denote such a $\phi_{\lambda_i}(t)$ with $\frac{1}{2}(k+1-\sqrt{-1}\lambda_i) = -m_i-n+i$ by $p_{m_i}(t)$. Thus $p_{m_i}(t)$ is a polynomial of degree m_i+n-i in $-\mathrm{sh}^2 t$. Then it follows from lemma 4.1 that $\phi_\lambda(a_T)$ is a polynomial of the form $\phi_\lambda(a_T) = c(-\mathrm{sh}^2 t_1)^{m_1}\ldots(-\mathrm{sh}^2 t_n)^{m_n}+$ terms of lower order (according to the lexicographical ordering of the n-tuples $(m_1,\ldots,m_n)$). This polynomial function we'll denote by $P_m(a_T)$ $(m \in N)$.

<u>DEFINITION 5.1</u>. Let $\mathbb{D}^W(G)$ be the set of all W-invariant differential operators on $\mathbb{R}^n$, regular in the interior of all Weyl chambers, and having all the P_m as eigenfunctions, that is $D \in \mathbb{D}^W(G)$ implies $DP_m = b(m)P_m$.

Clearly $\mathbb{D}^W(G)$ includes both $\delta(\mathbb{D}_0(G))$ and all polynomials in the Δ_j.

<u>LEMMA 5.2</u>. *Let* $D \in \mathbb{D}^W(G)$. *Let* $m = (m_1,\ldots,m_n) \in N$ *be the order of* D. *Then* D *is completely determined by its eigenvalues of* P_μ, $b(\mu)$, *with* $\mu \le m$.

<u>PROOF</u>. By the W-invariance of D, D can be written as a symmetric operator in $-\mathrm{sh}^2 t_1,\ldots,-\mathrm{sh}^2 t_n$. Let $-\mathrm{sh}^2 t_\sigma$ denote the vector $(-\mathrm{sh}^2 t_{\sigma(1)},\ldots,-\mathrm{sh}^2 t_{\sigma(n)})$ $(\sigma \in S_n)$. Then

$$D = \sum_{\mu\le m}\sum_{\sigma\in S_n} c_\mu(-\mathrm{sh}^2 t_\sigma)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\mu_{\sigma(1)}}\cdots\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\mu_{\sigma(n)}}.$$

We'll prove by complete induction with respect to μ that c_μ is completely determined by $b(\mu)$ $(\mu < m)$. We have $c_0 = b(0)$. It follows from $DP_\mu = b(\mu)P_\mu$ that

$$b(\mu)P_\mu = \sum_{\sigma\in S_n} c_\mu(-\mathrm{sh}^2 t_\sigma)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\mu_{\sigma(1)}}\cdots\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\mu_{\sigma(n)}}.P_\mu +$$

$$+\sum_{\substack{\nu<\mu\\ \neq}}\sum_{\tau\in S_n} c_\nu(-\mathrm{sh}^2 t_\tau)\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_1)}\right)^{\nu_{\tau(1)}}\cdots\left(\frac{\partial}{\partial(-\mathrm{sh}^2 t_n)}\right)^{\nu_{\tau(n)}}.P_\mu,$$

because the terms of the D with $\nu > \mu$ annihilate P_μ.

Hence

$$n!\beta_\mu c_\mu(-sh^2 t) = b(\mu)P_\mu - \sum_{\nu \lneq \mu} \sum_{\tau \in S_n} c_\nu(-sh^2 t_\tau)\left(\frac{\partial}{\partial(-sh^2 t_1)}\right)^{\nu_{\tau(1)}} \cdots \left(\frac{\partial}{\partial(-sh^2 t_n)}\right)^{\nu_{\tau(n)}} . P_\mu$$

where $\beta_\mu = \mu_1! \ldots \mu_n!$ times the coefficient of the term of order $(\mu_1,\ldots,\mu_n)$ in P_μ.

The lemma now follows by the induction hypothesis. □

LEMMA 5.2 immediately implies:

<u>LEMMA 5.3</u>. *Let* $D_1, D_2 \in \mathbb{D}^W(G)$. *Then* $D_1 D_2 = D_2 D_1$.

We have by definition $D \in \mathbb{D}^W(G) \Rightarrow D$ is W-invariant. W is the set of all maps s such that

$$s: (t_1,\ldots,t_n) \to (\varepsilon_1 t_{\sigma(1)},\ldots,\varepsilon_n t_{\sigma(n)}) \qquad \varepsilon_i = \pm 1 \quad \forall i,\ \sigma \in S_n.$$

This implies:

<u>LEMMA 5.4</u>. *Let* $D \in \mathbb{D}^W(G)$. *Suppose* D *is written in the form*

$$D = \sum_\mu c_\mu(t)\left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n}.$$

Then D *is invariant under the operations*

$$\begin{array}{lll} t_i & \to -t_i & \forall i \\ (t_1,\ldots,t_n) & \to (t_{\sigma(1)},\ldots,t_{\sigma(n)}) & \forall \sigma \in S_n. \end{array}$$

<u>LEMMA 5.5</u>. *Let* $D \in \mathbb{D}^W(G)$, *and let* d = degree D. *Then* D *can be written in the form*

$$(5.1) \qquad D = \sum_{\substack{\mu \\ \Sigma\mu_i = d}} c_\mu \left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n} + \text{l.o.}$$

(l.o. means lower order terms), where the c_μ *are constants.*

PROOF. Lemma 5.3 implies that D commutes with all the Δ_j, hence

(5.2) $$D\Delta_j - \Delta_j D = 0 \quad \text{for all } j.$$

We have

$$\Delta_j = S_j\left(\frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_n^2}\right) + \text{l.o.}$$

Let D be written in the form given by (5.1), only with $c_\mu = c_\mu(t)$. Now we use (5.2), in particular we use the fact that the terms of order $d+2j-1$ disappear. This yields:

$$(d+2j-1)^{\text{th}} \text{ order part of } \left[S_j\left(\frac{\partial^2}{\partial t_1^2}, \ldots, \frac{\partial^2}{\partial t_n^2}\right)\left\{\sum_{\substack{\mu \\ \Sigma\mu_i = d}} c_\mu(t)\left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n}\right\}\right] = 0.$$

Hence

$$\sum_{\substack{\nu \\ \Sigma\nu_i = d+2j-1}} \left(\sum_{p=1}^n \sum_{\pi \in V_p^{j-1}} \frac{\partial}{\partial t_p}\left(c_{\nu_1 - i_1(p,\pi), \ldots, \nu_n - i_n(p,\pi)}(t)\right) \cdot \left(\frac{\partial}{\partial t_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\nu_n}\right) = 0,$$

where we have defined:

- V_p^{j-1} := the set of all $(j-1)$-subsets of $\{1,\ldots,p-1,p+1,\ldots,n\}$

- $$i_q(p,\pi) := \begin{cases} 1 & \text{if } q = p \\ 2 & \text{if } p \in \pi \\ 0 & \text{else} \end{cases}$$

- $c_{j_1,\ldots,j_n} = o$ if one or more $j_i < 0$.

Hence we have to solve the system of equations

(5.3) $$\sum_{p=1}^n \sum_{\pi \in V_p^{j-1}} \frac{\partial}{\partial t_p}\left(c_{\nu_1 - i_1(p,\pi), \ldots, \nu_n - i_n(p,\pi)}(t)\right) = 0$$
$$\text{for all } 1 \le j \le n,\ \nu \text{ with } \Sigma\nu_i = d+2j-1.$$

We'll prove by complete induction with respect to the lexicographical ordering that (5.3) implies

$$(5.4) \qquad \frac{\partial}{\partial t_q} c_{\nu_1,\ldots,\nu_n}(t) = 0 \quad \forall q: 1 \le q \le n,\ \forall \nu: \Sigma\nu_i = d.$$

(Remember that $(\mu_1,\ldots,\mu_n) < (m_1,\ldots,m_n)$ iff. $\exists \ell$ such that $\mu_i = m_i$ $1 \le i \le \ell-1$ and $\mu_\ell < m_\ell$).

i. By taking $j = 1$ and $\nu_q = 1$, $\nu_i = 0$ for $i \neq q$ it is clear from (5.3) that $\frac{\partial}{\partial t_q} c_{0,\ldots,0}(t) = 0\ \forall q$.

ii. Let $(\ell_1,\ldots,\ell_n) = (0,\ldots,0,\ell_{p+1},\ldots,\ell_n)$ with $\ell_{p+1} \neq 0$, and assume that for all q

$$\frac{\partial}{\partial tq} c_{\ell_1',\ldots,\ell_n'}(t) = 0 \quad \text{if } (\ell_1',\ldots,\ell_n') < (\ell_1,\ldots,\ell_n)$$

(induction hypothesis).

a. Assume $1 \le q \le p$.
By taking $j = n-i+1$, $\nu_q = 1$, $\nu_i = 0$ if $1 \le i \le p$, $i \neq q$ and $\nu_1 = \ell_i+2$ if $i \ge p+1$ (5.3) becomes

$$\frac{\partial}{\partial t_q} c_{0,\ldots,0,\ell_{p+1},\ldots,\ell_n}(t) = 0.$$

b. Assume $q \ge p+1$.
By taking $j = n-q$, $\nu_i = 0$ if $1 \le i \le p$, $\nu_i = \ell_i$ if $p+1 \le i \le q-1$, $\nu_q = \ell_q+1$ and $\nu_i = \ell_i+2$ if $i \ge q+1$ (5.3) becomes

$$\frac{\partial}{\partial t_q} c_{0,\ldots,0,\ell_{p+1},\ldots,\ell_n}(t) = 0,$$

where we have used the induction hypothesis.

So it is proved that (5.3) implies (5.4). Hence $c_{\nu_1,\ldots,\nu_n}(t) =$ constant for all ν, so the lemma is proved. □

THEOREM 4. *Let* $D \in \mathbb{D}^W(G)$. *Then*

a. *D can be written as a polynomial in the* Δ_j.

b. *This expression is unique, that is if* $P_1(\Delta_1,\ldots,\Delta_n) = P_2(\Delta_1,\ldots,\Delta_n)$, *then* $P_1 \equiv P_2$.

PROOF. a. Let $D \in \mathbb{D}^W(G)$, and suppose D cannot be written as a polynomial in the Δ_j. Let d := degree D, and assume that d is minimal. According to lemma 5.5 we can write

$$D = \sum_{\substack{\mu \\ \Sigma\mu_i=d}} c_\mu \left(\frac{\partial}{\partial t_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\mu_n} + \text{l.o.}$$

Since D satisfies the symmetry relations of lemma 5.4, the d-th order part of D has to be a symmetric polynomial in $\frac{\partial^2}{\partial t_1^2},\ldots,\frac{\partial^2}{\partial t_n^2}$, and hence a polynomial in $S_1,\ldots,S_n$, where S_j is the j-th elementary symmetric polynomial in $\frac{\partial^2}{\partial t_1^2},\ldots,\frac{\partial^2}{\partial t_n^2}$. Thus we have

$$D = P(S_1,\ldots,S_n) + D',$$

where D' is an operator of degree < d. We also have $\Delta_j = S_j + \text{l.o.}$, so $S_j = \Delta_j + \text{l.o.}$ Hence

$$D = P(\Delta_1,\ldots,\Delta_n) + D'', \tag{5.5}$$

where D" is an operator of degree d" < d.

Since $D \in \mathbb{D}^W(G)$ and $P \in \mathbb{D}^W(G)$ (because all $\Delta_j \in \mathbb{D}^W(G)$) we have $D'' \in \mathbb{D}^W(G)$. Because d" < d, D" can be written as a polynomial in $\Delta_1,\ldots,\Delta_n$, and because of (5.5) this implies that D can be written as a polynomial in $\Delta_1,\ldots,\Delta_n$. This contradiction proves a.

b. It is sufficient to show: $Q(\Delta_1,\ldots,\Delta_n) = 0 \Rightarrow Q \equiv 0$, if Q is a polynomial. So, suppose $Q(\Delta_1,\ldots,\Delta_n) = 0$, and $Q \not\equiv 0$. So for some $e \in \mathbb{Z}$

$$Q(u) = \sum_{\substack{\mu \\ 2\mu_1+4\mu_2+\ldots+2n\mu_n\le e}} k_\mu u_1^{\mu_1} u_2^{\mu_2} \cdots u_n^{\mu_n},$$

where not for all μ with $2\mu_1+4\mu_2+\ldots+2n\mu_n = e$ we have $k_\mu = 0$. Taking $u_i = \Delta_i$, and using the fact that $\Delta_j = S_j\left(\frac{\partial^2}{\partial t_1^2},\ldots,\frac{\partial^2}{\partial t_n^2}\right) + \text{l.o.}$ we obtain

$$0 = Q(\Delta_1,\dots,\Delta_n) = \sum_{\substack{\mu \\ 2\mu_1+\dots+2n\mu_n \le e}} k_\mu (S_1+\text{l.o.})^{\mu_1}(S_2+\text{l.o.})^{\mu_2}\dots(S_n+\text{l.o.})^{\mu_n}$$

$$= \sum_{\substack{\mu \\ 2\mu_1+\dots+2n\mu_n = e}} k_\mu \left(S_1\left(\frac{\partial^2}{\partial t_1^2},\dots,\frac{\partial^2}{\partial t_n^2}\right)\right)^{\mu_1}\dots\left(S_n\left(\frac{\partial^2}{\partial t_1^2},\dots,\frac{\partial^2}{\partial t_n^2}\right)\right)^{\mu_n} + \text{l.o.}$$

Hence, the e-th order term of the above expression must be 0. But this is a combination of elementary symmetric polynomials, and this combination can only be 0 if all coefficients are 0, hence

$$k_\mu = 0 \quad \forall\mu\colon\ 2\mu_1+4\mu_2+\dots+2n\mu_n = e,$$

which is a contradiction, so $Q \equiv 0$. □

Because of theorem 4 we have proved the second theorem of BEREZIN and KARPELEVIČ [1].

<u>THEOREM 5</u>. *Let* $G = SU(n,n+k;\mathbb{C})$. *The operators* $\Delta_j = \omega^{-1} S_j(L_1,\dots,L_n)\circ\omega$ $(1 \le j \le n)$, where S_j = j-th *elementary symmetric polynomial and* $L_i = \frac{\partial^2}{\partial t_i^2} + 2(k \coth t_i + \coth 2t_i)\frac{\partial}{\partial t_i}$, *form a system of generators for* $\delta(\mathbb{D}_0(G))$.

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